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考虑纵向摩阻时弹性地基梁的弯曲

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摘 要:考虑地基纵向摩擦力的影响,建立了广义 Winkler 型地基梁的平衡方程. 假定纵向摩擦力与梁底面的纵向位移成正比,引入广义剪力,得到梁的位移型平衡方程. 将位移及荷载展开为带附加项的 Fourier 级数,利用平衡方程和边界条件对弹性地基梁的一般弯曲进行分析. 分析表明,梁的位移和内力均受到纵向摩擦力的影响,并且随着纵向反力系数和梁截面高度的增大而增大;梁的最大挠度、转角、弯矩及剪力随着地基纵向反力系数的增大而减小;梁的轴向位移和轴力则随着地基纵向反力系数的增大而增大.

关键词:纵向摩擦;梁;地基;Fourier 级数

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弹性地基梁是结构基础工程中的主要构件之一,准确合理地对其进行计算具有重要的意义. 地基计算模型先后提出了 Winkler 模型、弹性连续介质模型、双参数模型和有限压缩层模型等,其中 Winkler 模型因其简单实用仍然得到广泛的应用. 传统分析方法中一般假设梁与地基为光滑接触,通过综合分析梁、地基及两者之间的法向协调变形关系对弹性地基梁进行求解[1-2]. 事实上,由于梁与地基之间为非光滑接触且接触面较为粗糙,另外,混凝土材料与土的变形模量有显著差异,使得梁与地基的接触面性质非常复杂,客观上就存在着纵向摩擦力,而这种接触面的纵向摩阻通常不容忽略,否则可能会对计算结果造成较大误差.

谈至明[3]首先探讨了考虑纵向摩阻力时地基梁的解,之后一些学者也开始对该问题进行探索[4-9]. 通过分别假定摩擦力与梁的底面位移成正比,或与接触压力成正比,或假设梁底地基纵向摩擦力为线性分布,或假定纵向摩阻力为定值,分别通过直接求解特征方程、分步计算、伽辽金法、幂级数法、微分算子级数法及有限元法进行了分析.

本文通过引入一个广义剪力,以梁轴线处的轴向位移和法向挠度为求解的两个基本未知量,利用 Fourier 级数研究了考虑纵向摩阻时的 Winkler 型弹性地基梁的一般弯曲.

1 理论分析

对图 1 所示的弹性地基梁,假定地基为广义 Winkler 地基,即横向反力与梁的挠度成正比,纵向反力与梁底面的纵向位移成正比,于是

$$p_v = k_v w, \quad p_h = k_h \left(u - \frac{h}{2}\frac{\mathrm{d}w}{\mathrm{d}x}\right) \tag{1}$$

其中, k_v 、k_h 分别为地基横向和纵向反力系数, u 、w 分别为梁轴线处的轴向位移、挠度.

不计轴向力产生的附加横向荷载,由所取梁段的平衡,得

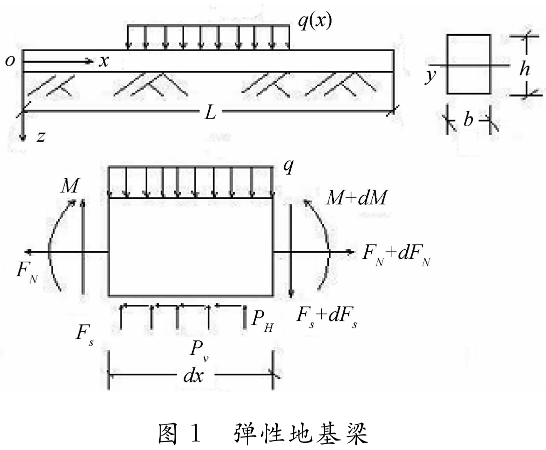

图 1 弹性地基梁

Fig. 1 A beam on the elastic foundation

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$$\frac{\mathrm{d}F_N}{\mathrm{d}x}=p_h,\quad \frac{\mathrm{d}F_S}{\mathrm{d}x}=-q+p_v,\quad \frac{\mathrm{d}M}{\mathrm{d}x}=F_S+\frac{h}{2}p_h. \tag{2}$$

让 x 轴通过截面形心，y、z 轴为截面的形心主轴，于是

$$\begin{aligned} F_N&=\int_A\sigma\mathrm{d}A=\int_A E\left(\frac{\mathrm{d}u}{\mathrm{d}x}-z\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)\mathrm{d}A=EA\frac{\mathrm{d}u}{\mathrm{d}x}\\ M&=\int_A\sigma z\mathrm{d}A=\int_A E\left(\frac{\mathrm{d}u}{\mathrm{d}x}-z\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)z\mathrm{d}A=-EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\end{aligned} \tag{3}$$

定义如下的广义剪力

$$V=F_S+\frac{h}{2}p_h \tag{4}$$

则由式(2)、(3)知

$$V=-EI\frac{\mathrm{d}^3w}{\mathrm{d}x^3} \tag{5}$$

上式中，EA 为梁的抗拉刚度，EI 为梁的抗弯刚度.

将式(3)、(5)代入式(2)中，则可得到以位移 u 和 w 表示的梁的平衡方程，即

$$EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-k_hu+\frac{h}{2}k_h\frac{\mathrm{d}w}{\mathrm{d}x}=0,\quad EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4}-\left(\frac{h}{2}\right)^2k_h\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+k_vw+\frac{h}{2}k_h\frac{\mathrm{d}u}{\mathrm{d}x}=q. \tag{6}$$

设梁的位移及作用在梁上的荷载可展为如下的 Fourier 级数[10]：

$$\begin{aligned} u&=\sum_{m=1}u_m\sin\left(\frac{m\pi}{L}x\right)+\left(1-\frac{x}{L}\right)a_0+\frac{x}{L}a_1,\\ w&=\frac{w_0}{2}+\sum_{m=0}w_m\cos\left(\frac{m\pi}{L}x\right)+\frac{L}{2}\left[2\frac{x}{L}-\left(\frac{x}{L}\right)^2\right]b_0+\frac{L}{2}\left(\frac{x}{L}\right)^2b_1,\\ q&=\frac{q_0}{2}+\sum_{m=1}q_m\cos\left(\frac{m\pi}{L}x\right),\quad q_m=\frac{2}{L}\int_0^L q(x)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x.\end{aligned} \tag{7}$$

其中的 u_m、w_m 及 a_0、a_1、b_0、b_1 为待定常数，级数中所加的附加项是为了保证 u 和 w 能连续求导.

将式(7)代入式(6)中，再把 a_0、a_1、b_0、b_1 的系数展开为相应的 Fourier 正弦级数或余弦级数，通过比较系数得

$$\begin{aligned}&\left[EA\left(\frac{m\pi}{L}\right)^2+k_h\right]u_m+\frac{hk_h}{2}\frac{m\pi}{L}w_m\\ &\quad+\frac{2k_h}{m\pi}a_0-\frac{2(-1)^mk_h}{m\pi}a_1-\frac{hk_h}{m\pi}b_0+\frac{(-1)^mhk_h}{m\pi}b_1=0\quad(m=1,2,3,\cdots)\end{aligned} \tag{8}$$

$$\frac{k_v}{2}w_0-\frac{hk_h}{2L}a_0+\frac{hk_h}{2L}a_1+\left[\frac{k_vL}{3}+\left(\frac{h}{2}\right)^2\frac{k_h}{L}\right]b_0+\left[\frac{k_vL}{6}-\left(\frac{h}{2}\right)^2\frac{k_h}{L}\right]b_1=\frac{1}{2}q_0\quad(m=0) \tag{9}$$

$$\begin{aligned}&\frac{hk_h}{2}\frac{m\pi}{L}u_m+\left[EI\left(\frac{m\pi}{L}\right)^4+k_v+\left(\frac{h}{2}\right)^2k_h\left(\frac{m\pi}{L}\right)^2\right]w_m\\ &\quad-\frac{2k_v}{L}\left(\frac{L}{m\pi}\right)^2b_0+\frac{2(-1)^mk_v}{L}\left(\frac{L}{m\pi}\right)^2b_1=q_m\quad(m=1,2,3,\cdots)\end{aligned} \tag{10}$$

将式(7)代入内力表达式(3)、(5)中，则有

$$\begin{aligned} F_N&=EA\left[\sum_{m=1}u_m\frac{m\pi}{L}\cos\left(\frac{m\pi}{L}x\right)-\frac{1}{L}a_0+\frac{1}{L}a_1\right],\\ M&=EI\left[\sum_{m=1}w_m\left(\frac{m\pi}{L}\right)^2\cos\left(\frac{m\pi}{L}x\right)+\frac{1}{L}b_0-\frac{1}{L}b_1\right],\\ V&=-EI\sum_{m=1}w_m\left(\frac{m\pi}{L}\right)^3\sin\left(\frac{m\pi}{L}x\right).\end{aligned} \tag{11}$$

两端自由弹性地基梁，其边界条件如下：

$$x=0,L:\quad F_N=0,\quad M=0,\quad V=0. \tag{12}$$

即

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0,L}=0,\quad \left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=0,L}=0,\quad \left.\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right|_{x=0,L}=0. \tag{13}$$

由式(11)可知，式(12)或(13)中的第三个边界条件已满足；由式(13)中的第一、第二个边界条件得

$$\begin{aligned}&\sum_{m=1} m\pi u_m - a_0 + a_1 = 0,\\&\sum_{m=1} (-1)^m m\pi u_m - a_0 + a_1 = 0,\\&\sum_{m=1} (m\pi)^2 w_m + Lb_0 - Lb_1 = 0,\\&\sum_{m=1} (-1)^m (m\pi)^2 w_m + Lb_0 - Lb_1 = 0.\end{aligned} \tag{14}$$

式(8)、(9)、(10)和(14)即为本文利用 Fourier 级数法所得到的控制方程组. 若级数取到前 M 项，则待定系数 u_m 、w_m 及 a_0 、a_1 、b_0 、b_1 总共有 $M+(M+1)+4$ 个，代数方程组(8)、(9)、(10)和(14)中也共有 $M+(M+1)+4$ 个代数方程，因此问题可解.

2　算例及分析

设梁的长度 $L=20$ m ，截面尺寸 $h=1.5$ m，$b=0.5$ m ，弹性模量 $E=20\times10^9$ Pa ，地基反力系数 $k_v=3\times10^7\,\mathrm{N/m^2}$，$k_h=0$，$3\times10^8\,\mathrm{N/m^2}$，$15\times10^8\,\mathrm{N/m^2}$ ，荷载为均布荷载 $q_1=0.4$ kN/m 及集中力 $q_2=400\delta(x-L/2)$ kN ，这里 $\delta(x)$ 为 Diracδ 函数. 计算结果见图 2～7.

计算结果表明，由于在梁和地基之间所存在的纵向摩擦力的影响，梁的挠度、转角、弯矩及剪力这些横向力学量较不考虑地基摩阻时有所减小，且其减小的程度随地基纵向反力系数的增大而加大；而梁的轴向位移和轴力这些轴向力学量则随着地基纵向反力系数的增大而增大.

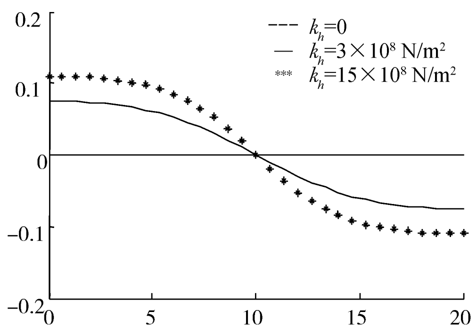

图 2　轴向位移变化规律

Fig. 2　The axial displacement of beam

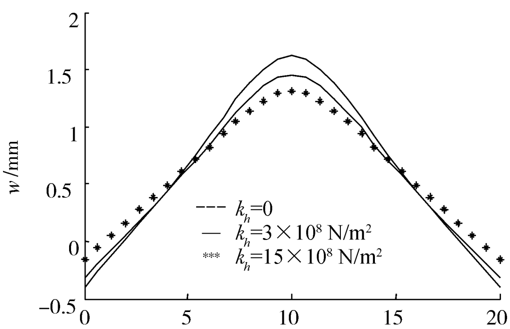

图 3　挠度变化规律

Fig. 3　The deflections of beam

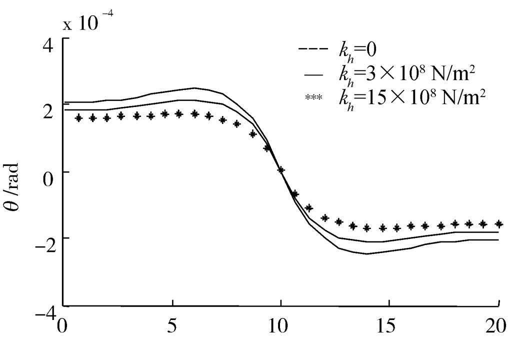

图 4　转角变化规律

Fig. 4　The slopes of beam

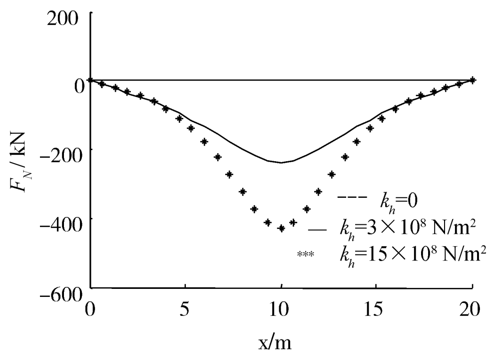

图 5　轴力变化规律

Fig. 5　The axial forces of beam

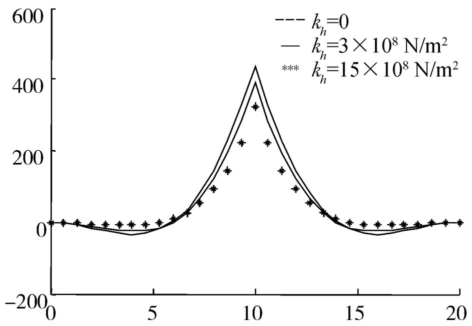

图 6　弯矩变化规律

Fig. 6　Bending moments of beam

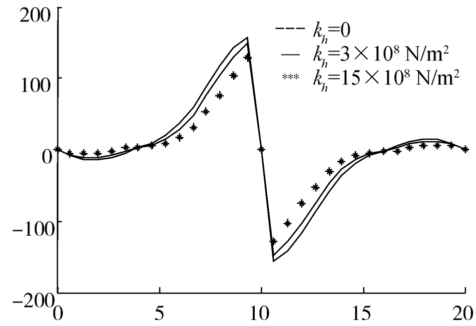

图 7　剪力变化规律

Fig. 7　Shear forces of beam

由轴向位移和转角的计算结果,综合纵向摩擦力的计算公式,可知纵向摩擦力的分布规律是较为复杂的,不是简单的线性分布或定值.轴线处的轴向位移分量和梁与地基接触面的纵向位移分量具有同一数量级(见式(1)),因此忽略截面中心轴向位移对纵向摩擦力的影响值得商榷.

3　结　论

(1)利用 Fourier 级数求解弹性地基梁是一种有效的方法.

(2)当考虑地基纵向摩阻时,梁的剪力与挠度的 3 阶导数不再成正比例关系.

(3)纵向摩擦力对梁的影响随着纵向反力系数和梁截面高度的增大而增大.

(4)纵向摩擦力的分布规律尚无定论,究竟哪一种规律更接近实际情况还需实验验证.

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Evaluation of asphalt mortar with High Viscosity at Low Temperature

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Abstract: Properties of asphalt mortar with different asphalt and mineral powder at low temperature are studied in this paper. Influence of material composition and ratio of filler on bending and tensile properties of asphalt mortar is analyzed. The results show that the type of mineral powder and bitumen affects the properties of asphalt mortar obviously. Asphalt mortar composed of higher viscosity bitumen and mineral powder with larger specific surface area have smaller bending stiffness modulus, bigger fracture energy and better properties at low temperature. Results of bending testing at -10℃ show that different asphalt mortar has its own optimum ratio of filler at low temperatures, ranging between 1.2～1.4. Also. In order to get the better low temperature properties, the ratio of filler to bitumen should be smaller than 1.4.

Key words: *pavement engineering*; *asphalt mortar with high viscosity*; *bending properties at low temperature*; *tensile properties*

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The bending of beam on the elastic foundation considering longitudinal frictional resistances

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Abstract: The equilibrium equations of beam on the general Winkler elastic foundation are established considering longitudinal friction. Assuming that the longitudinal friction is proportional to the beam's base longitudinal displacement, the equations of equilibrium in the forms of displacement can be obtained after introducing a general shearing force. By expanding the displacements and loads to Fourier series with additional terms, the bending problems of elastic foundation beam are analyzed by using the equilibrium equations and boundary conditions. The results indicate that the beam's displacements and internal forces are related to the longitudinal friction and the influences of friction are magnified with the longitudinal force coefficient and height of cross-section of beam. As the longitudinal force coefficient increased, the maximums of deflection, slope, bending moment and shear force decreased but the maximums of axial displacement and axial force increased.

Key words: *longitudinal friction*; *beam*; *foundation*; *Fourier series*

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